

Unification of Gravity and Electromagnetism and Cosmology

Partha Ghose*

The National Academy of Sciences, India,
5 Lajpatrai Road, Allahabad 211002, India

Abstract

It is first argued that radiation by a uniformly accelerated charge in flat space-time indicates the need for a unified geometric theory of gravity and electromagnetism. Such a theory, based on a metric-affine U_4 manifold, is constructed with the torsion pseudo-vector Γ_μ linking gravity and electromagnetism. This conceptually simple extension results in (i) Einstein's equations being modified by a vacuum energy $\Gamma_\mu\Gamma_\nu$ and a scalar field $\Gamma = \Gamma^\mu\Gamma_\mu$ whose zero-mode is a cosmological constant Λ representing 'dark energy', (ii) most of the salient features of 'dark matter'-like phenomena, (iii) a modified electrodynamics satisfying Heaviside duality, (iv) a finite and small Casimir Effect, and at the same time, (v) the empirical Schuster-Blackett-Wilson relation for the amazingly universal gyromagnetic ratio of slowly rotating, neutral astrophysical bodies.

1 The Equivalence Principle

The Equivalence Principle, together with the requirement of covariance of the laws of physics under the most general coordinate transformations, constitutes the physical basis of Einstein's General Theory of Relativity. It consists of two statements:

Universality of Free Fall or Weak Equivalence Principle (WEP)

All test bodies fall in a gravitational field with the same acceleration regardless of their mass or internal composition.

This is Galileo's law of falling bodies which is incorporated in Newton's theory of gravity through the equality of gravitational and inertial mass which has been confirmed to high precision by the Eötvös experiment. Einstein enunciated a stronger form of the principle which states that the motion of a test particle in a locally homogeneous gravitational field is physically indistinguishable from that of the particle at rest in a uniformly accelerating coordinate system. It has also been stated in the form [1]:

Einstein's Equivalence Principle (EEP)

For every infinitesimally small world region in which space-time variations of gravity can be neglected, there always exists a coordinate system in which gravitation has no influence either on the motion of test particles or any other physical process.

*partha.ghose@gmail.com

This means that a homogeneous gravitational field in an infinitesimal world region \mathcal{R} (strictly speaking only at the centre of mass of an Einstein chamber or lift) can be ‘transformed away’ relative to an observer in free fall in that field. However, for practical tests of the principle it is often difficult to specify the region \mathcal{R} over which the gravitational field is strictly uniform and can be transformed away. This is why the following statement is preferred by some [2]:

For every infinitesimally small world region in which space-time variations of gravity can be neglected, there always exists a coordinate frame in which, from the point of view of the co-moving observer, gravitation appears to have no influence on the motion of test particles, or on any other natural phenomenon, but only as to local measurements that ignore the motion of test objects with respect to the source(s) of the field.

In order to have a clear understanding of Einstein’s own statement of the principle, let us look at how he actually argued in his 1916 paper [3]. Let us consider a Galilean frame K relative to which test particles of different masses are at rest or in uniform motion in a straight line in a world region \mathcal{R} which is far removed from other masses, i.e. free of any gravitational field or any other force, and is flat and Minkowskian. Let K' be a second co-ordinate system which has an *arbitrary uniform acceleration* relative to K . Relative to K' all the particles experience the same acceleration in the opposite direction *independent of their material compositions and physical conditions*.

To an observer O' at rest relative to K' , K' has no acceleration. Can O' conclude that she is in an actually accelerated reference system? The answer is ‘no’ because the common accelerated motion of the freely moving masses relative to K' can be equally well *conceptualized* by saying that K' is not accelerated and that in the world region \mathcal{R} there is a homogeneous gravitational field which generates the accelerated motion relative to K' . This conception is feasible because we know that the gravitational field has the remarkable property of imparting the same acceleration to all bodies. Hence, from the physical point of view *the two frames K and K' can both be regarded as Galilean with the same legitimacy*, i.e., they are to be treated as equivalent systems of reference for a description of physical phenomena.

Thus, acceleration is shorn of its absolute character and, like uniform velocity in a straight line, rendered relative. In the full theory, space-time is no longer Minkowskian and assumes a dynamical nature whose geometry is determined by the actual distribution of masses, and forces disappear altogether from the theory. Minkowski space-times linger in the theory only as tangent planes to a pseudo-Riemannian manifold. This is a very important point that will be exploited in what follows.

According to Einstein, the physical equivalence of K and K' also follows from an epistemological argument due to Mach (a component of what is known as Mach’s Principle). In Newtonian physics a uniform velocity in a straight line is relative and cannot be physically distinguished from rest, but acceleration is absolute. This is because space against which acceleration can be measured is absolute. But absolute space is not observable. Newton was aware of this problem as his comments on the famous rotating bucket experiment clearly show. Mach put forward the epistemological principle that something can be said to be the cause of some effect only when that thing is an observable fact of experience. In other words, the law of causality can be given a meaningful statement about the world of experience only when observable facts alone appear as causes and effects. Since absolute space is not an observable fact of experience, *there is no observable ground on which K and K' can be distinguished*. Hence both can be treated as inertial or Galilean frames with equal legitimacy, and what appears to be a uniform gravitational field relative to K' disappears relative to K . With this clearly in mind, let us now pass on to the case of charged particles.

2 EEP and Electrically Charged Particles

According to standard classical electrodynamics texts, an accelerated charge invariably radiates electromagnetic waves. However, there is a “perpetual problem” as to whether a uniformly accelerated charge radiates in flat space-time [4]. In the non-relativistic limit the equation of motion of a charged particle is

$$m\vec{a} = \vec{F}_{ext} + \vec{F}_{react} \quad (1)$$

where the first term is an external force on the particle due to a gravitational or electromagnetic field and the second term is the radiation reaction force given by

$$\vec{F}_{react} = \frac{q^2}{6\pi\epsilon_0 c^3} \dot{\vec{a}} + \mathcal{O}(v/c). \quad (2)$$

Since $\dot{\vec{a}} = 0$ for uniform acceleration, $\vec{F}_{react} = 0$ and it has been argued by many that this means there is no radiation [1, 4, 5, 6]. Now, this view is consistent with EEP because of the following reason. Suppose there are two particles of the same mass but only one of them is charged. Then, if they are accelerated by the same gravitational force, the charged particle will radiate but not the neutral particle. Consequently, the charged particle will lose energy and decelerate compared to the neutral particle. That will violate WEP (and hence EEP) which requires both of them to have the same acceleration. One way to reconcile standard electrodynamics with EEP is to argue that the equations of standard electrodynamics actually do not imply that uniformly accelerated charges radiate, as briefly shown above. However, this claim has been in a mire of controversy [7]. The radiation reaction or self-force, also known as the Lorentz-Abraham-Dirac (LAD) force, is known to have pathological solutions like *pre-acceleration* and *run-away* solutions. A recent review of the field will be found in Hammond [8].

In 1955 Bondi and Gold claimed that within the context of General Relativity a static charge in a static gravitational field cannot radiate energy, thus appearing to violate a particular version of the equivalence principle [9]. They resolved this paradox by showing that hyperbolic motion requires a homogeneous gravitational field of infinite extension and that such a field does not exist in nature. The ensuing debate by DeWitt and Brehme [10], Fulton and Rohrlich [11], Boulware [12], Parrott [13] and others has been reviewed by Grøn [14] and Lyle [15], and the matter still remains unresolved. One of the main concerns is whether the controversial radiation is consistent with EEP.

Let us turn the question around and ask: under what conditions can radiation by a uniformly accelerated charge in flat space-time be reconciled with EEP *without requiring the controversial radiation reaction force*? The first point to note is that since General Relativity requires acceleration to be relative, and only an accelerated charge can radiate, *electromagnetic radiation must also be relativized*. In other words, radiation and gravity must be generated and also transformed away by the same coordinate transformation. *This requires some sort of unification of electrodynamics and gravity*.

To elaborate the point consider a test particle A with charge in an infinitesimally small world region \mathcal{R} which is sufficiently far removed from all other bodies and hence flat and Minkowskian. Let K be a Galilean frame relative to which A is at rest. Relative to another frame K' which has an arbitrary uniform acceleration relative to K , A obviously has a uniform acceleration and radiates. As we have seen, an observer O' at rest relative to K' can legitimately consider K' to be at rest and Galilean, and *conceptualize* the observed radiation by a uniformly accelerated A by saying that in addition to the usual homogeneous gravitational field, the world region \mathcal{R} possesses another homogeneous gravitational field *of electromagnetic origin* which keeps the acceleration of A uniform as it radiates. The radiation and the associated gravitational acceleration must be such as to disappear relative to K .

Now consider the Poynting theorem

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\vec{E} \times \vec{B}) = 0 \quad (3)$$

where $\rho = \frac{1}{2}[\epsilon_0 \vec{E} \cdot \vec{E} + \frac{1}{\mu_0} \vec{B} \cdot \vec{B}]$ is the energy density of the electromagnetic field in \mathcal{R} and $\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$ is the Poynting vector. This means

$$\text{div} \vec{S} = -\frac{\partial \rho}{\partial t} \equiv P, \quad (4)$$

and $\text{div} \vec{S} > 0$ only if $P > 0$. Hence, a positive ρ must exist for radiation to occur. Let us now consider the special case of a single charged body A which is both stationary and accelerating. For example, a charged body in a homogeneous gravitational field can be held fixed at a point relative to a Galilean frame K' , and yet it is also subject to a proper gravitational acceleration. If we set $\vec{v}_A = 0$ relative to K' , we get from Poynting's theorem

$$\text{div} \vec{S} = -\frac{\partial \rho}{\partial t} = 0 \quad (5)$$

in a static condition. Hence, such a charged body does not radiate relative to O' . However, the body will be accelerated relative to K , and hence will radiate relative to it. Thus, like gravitation, *radiation must be relativized* once the principle of relativity is extended to all coordinate frames no matter how they move relative to one another. Some of the debate in the literature stems from a failure to see this point.

It must be emphasized that one is not arguing for an equivalence principle for electromagnetism as a whole because, unlike mass, electric charge q can be positive and negative, and the q/m ratio is not universal. The issue is whether radiation by a uniformly accelerated charge requires a unification of gravity and electromagnetism.

Before concluding this section, it would be worthwhile to recall how Einstein summarized the situation regarding his Equivalence Principle and his theory of gravity [21]:

In fact, through this conception we arrive at the unity of the nature of inertia and gravitation. For according to our way of looking at it, the same masses may appear to be either under the action of inertia alone (with respect to K) or under the combined action of inertia and gravitation (with respect to K').

The above analysis shows that it would be possible to make a similar statement about EEP and a unified theory of gravity and electromagnetism, if it were to exist, namely that through the above considerations one

would arrive at the unity of the nature of inertia, gravitation and electromagnetism. For according to this way of looking at it, the same bodies may appear to be either under the action of inertia alone or under the combined action of inertia and unified gravity and electromagnetism.

We will now proceed to construct such a unified theory.

3 Unification of Gravity and Electromagnetism

A proper understanding of how gravity and electromagnetism can be simultaneously generated or transformed away by a general coordinate transformation requires a proper unified theory of the

two fields at the classical level. Goenner has written a history of such unified theories [16]. They were abandoned in the past because of several reasons. First, they attempted to solve ‘all problems regarding the elementary particles of matter with the help of classical fields which are everywhere regular (free of singularities)’ [17], which proved unsuccessful. Second, quantum mechanics and nuclear physics were discovered, and the hope of achieving any fundamental understanding in purely classical terms was given up. Third, the electromagnetic and gravitational coupling strengths are enormously different, and any symmetry between these fields must be badly broken, but the idea of broken symmetries that can be restored in some regime was then unknown.

We will take the point of view that unlike the nuclear interactions, gravity and electromagnetism are the only two long range fields that are already very well described in classical terms, and hence a search for their unity in purely classical terms without being ambitious to solve ‘all problems regarding the elementary particles of matter with the help of classical fields which are everywhere regular (free of singularities)’ would be worthwhile, particularly because no attempt to quantize gravity has been successful so far. The *immediate motivation* for the search for unity, however, comes from the requirement of EEP in the presence of charged particles as discussed in the previous section.

After Weyl’s and Kaluza’s attempts at unification, it was Eddington [18], supported by Einstein [19, 20], who first proposed to replace the metric as a fundamental concept by a non-symmetric affine connection Γ and a non-symmetric metric g which can then be split into a symmetric and an anti-symmetric part. Just as passing beyond Euclidean geometry gravitation makes its appearance, so going beyond Riemannian geometry electromagnetism appears *naturally* as the anti-symmetric part of the metric *without requiring any higher dimensional space*. Let $\mathcal{M}(\Gamma, g)$ be a smooth U_4 manifold with signature $(-, +, +, +)$ and endowed with a non-symmetric linear connection Γ and a non-symmetric metric g . If one splits the non-symmetric connection into a symmetric and an anti-symmetric part,

$$\Gamma_{\mu\nu}^{\lambda} = \Gamma_{(\mu\nu)}^{\lambda} + \Gamma_{[\mu\nu]}^{\lambda}, \quad (6)$$

$$\Gamma_{(\mu\nu)}^{\lambda} = \frac{1}{2} (\Gamma_{\mu\nu}^{\lambda} + \Gamma_{\nu\mu}^{\lambda}), \quad (7)$$

$$\Gamma_{[\mu\nu]}^{\lambda} = \frac{1}{2} (\Gamma_{\mu\nu}^{\lambda} - \Gamma_{\nu\mu}^{\lambda}) \equiv Q_{\mu\nu}^{\lambda}, \quad (8)$$

then $\Gamma_{[\mu\nu]}^{\lambda}$ is called the Cartan torsion tensor and $\Gamma_{\mu} = \Gamma_{[\mu\lambda]}^{\lambda}$ the torsion pseudovector. One can also split the non-symmetric metric $g^{\mu\nu}$ into the symmetric and anti-symmetric tensor densities

$$s^{\mu\nu} = \frac{1}{2} \sqrt{-g} (g^{\mu\nu} + g^{\nu\mu}) \equiv \frac{1}{2} (\bar{g}^{\mu\nu} + \bar{g}^{\nu\mu}) = \frac{1}{2} \sqrt{-g} g^{(\mu\nu)}, \quad (9)$$

$$a^{\mu\nu} = \frac{1}{2} \sqrt{-g} (g^{\mu\nu} - g^{\nu\mu}) \equiv \frac{1}{2} (\bar{g}^{\mu\nu} - \bar{g}^{\nu\mu}) = \frac{1}{2} \sqrt{-g} g^{[\mu\nu]}. \quad (10)$$

To restrict the number of possible covariant terms in a non-symmetric theory [21], Einstein and Kaufman imposed *transposition invariance* and Λ -*transformation invariance* on the theory. Let us first note the definitions of these symmetries.

Λ -transformation or projective symmetry

Define the transformations

$$\begin{aligned} \Gamma_{\mu\nu}^{\lambda*} &= \Gamma_{\mu\nu}^{\lambda} + \delta_{\mu}^{\lambda} \lambda_{,\nu}, \\ g^{\mu\nu*} &= g^{\mu\nu}, \end{aligned} \quad (11)$$

where λ is an arbitrary function of the coordinates. Then the contracted curvature tensor

$$E_{\mu\nu} = \Gamma_{\mu\nu,\lambda}^\lambda - \Gamma_{\mu\lambda,\nu}^\lambda + \Gamma_{\mu\nu}^\xi \Gamma_{\xi\lambda}^\lambda - \Gamma_{\mu\lambda}^\xi \Gamma_{\xi\nu}^\lambda \quad (12)$$

which is the generalization of the Ricci tensor $R_{\mu\nu}$ to the non-symmetric theory, is λ -transformation invariant. What this means is that a theory characterized by $E_{\mu\nu}$ cannot determine the Γ -field completely but only up to an arbitrary function λ . Hence, in such a theory, Γ and Γ^* represent the same field. Further, this λ -transformation produces a non-symmetric Γ^* from a Γ that is symmetric or anti-symmetric in the lower indices. Hence, the symmetry condition for Γ loses objective significance. This sets the ground for a genuine unification of gravity and electromagnetism, the former determined by the symmetric part and the latter by the antisymmetric part of the action.

Transposition symmetry

Let $\tilde{\Gamma}_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda$ and $\tilde{g}_{\mu\nu} = g_{\nu\mu}$. Then terms that are invariant under the simultaneous replacements of $\Gamma_{\mu\nu}^\lambda$ and $g_{\mu\nu}$ by $\tilde{\Gamma}_{\nu\mu}^\lambda$ and $\tilde{g}_{\nu\mu}$ are called transposition invariant. For example, the tensor $E_{\mu\nu}$ (12) is not transposition invariant because it is transposed to

$$\tilde{E}_{\nu\mu} = \Gamma_{\nu\mu,\lambda}^\lambda - \Gamma_{\lambda\mu,\nu}^\lambda + \Gamma_{\nu\mu}^\xi \Gamma_{\lambda\xi}^\lambda - \Gamma_{\lambda\mu}^\xi \Gamma_{\nu\xi}^\lambda. \quad (13)$$

However, if new quantities

$$\begin{aligned} U^\lambda_{\mu\nu} &= \Gamma^\lambda_{\mu\nu} - \Gamma^\alpha_{\mu\alpha} \delta^\lambda_\nu, & U^\lambda_{\mu\lambda} &= -3\Gamma^\lambda_{\mu\lambda} \\ \Gamma^\lambda_{\mu\nu} &= U^\lambda_{\mu\nu} - \frac{1}{3} U^\alpha_{\mu\alpha} \delta^\lambda_\nu, \end{aligned} \quad (14)$$

are introduced, then the contracted curvature tensor expressed in terms of the $U^\lambda_{\mu\nu}$,

$$E_{\mu\nu}(U) = U^\lambda_{\mu\nu,\lambda} - U^\lambda_{\mu\beta} U^\beta_{\lambda\nu} + \frac{1}{3} U^\lambda_{\mu\lambda} U^\beta_{\beta\nu}, \quad (15)$$

is transposition invariant. However, as Pauli pointed out, EEP is entirely missing from a unified theory based on $E_{\mu\nu}(U)$ [17]. He also observed that such theories ‘are in disagreement with the principle that only irreducible quantities should be used in field theories’ and no cogent mathematical reasons were given as to why the decomposition of the quantities $(E(U), g, \Gamma)$ used in the theory do not occur.

A variant of Bose’s 1953 theory

In 1953 S. N. Bose [22] proposed a variation of Einstein’s idea in which, unlike Einstein, he did not set the torsion pseudovector Γ_μ to zero and also used irreducible quantities. We will develop a variant of Bose’s theory with additional λ -transformation symmetry or ‘projective symmetry’ as it is called now. A matter Lagrangian $\mathcal{L}_m(\psi, g, \Gamma)$ obtained by minimally coupling matter fields ψ to the connection is not generally projective invariant. Hence, once the matter Lagrangian is introduced, projective symmetry is explicitly broken. We will first formulate a projective and transposition invariant theory to unify gravity and electromagnetism and then break projective invariance explicitly. ‘Spacetime’ will emerge with its geometrical properties determined by the stress-energy of matter (as in the Einstein theory) as well as by the torsion tensor $\Gamma_\mu \Gamma_\nu$. Instead of introducing the quantities $U^\lambda_{\mu\nu}$ to have transposition invariance, Bose achieved transposition invariance by writing the invariant action in the form

$$2\kappa I = \frac{1}{2} \left[\bar{g}^{\mu\nu} E_{\mu\nu} + \bar{g}^{\nu\mu} \tilde{E}_{\nu\mu} \right] + a s^{\mu\nu} \Gamma_\mu \Gamma_\nu + b a^{\mu\nu} (\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}) \quad (16)$$

where $\kappa = 8\pi G/c^4$, and a, b are arbitrary dimensionless constants, but this action is not projective invariant unless $a = 0$ because

$$\begin{aligned}\Gamma_{\mu\lambda}^{\lambda*} &= \Gamma_{\mu\lambda}^\lambda + \lambda_{,\mu}, \\ \Gamma_{[\mu\lambda]}^{\lambda*} &= \Gamma_\mu^* = \Gamma_\mu + \lambda_{,\mu},\end{aligned}\tag{17}$$

and hence $\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}$ is projective invariant but not $\Gamma_\mu \Gamma_\nu$. As shown in the Appendix (Part 1), the action (16) can be expressed in terms of the Ricci tensor

$$R_{\mu\nu} = \Gamma_{(\mu\nu),\lambda}^\lambda - \Gamma_{(\mu\lambda),\nu}^\lambda + \Gamma_{(\mu\nu)}^\xi \Gamma_{(\xi\lambda)}^\lambda - \Gamma_{(\mu\lambda)}^\xi \Gamma_{(\xi\nu)}^\lambda\tag{18}$$

and its antisymmetric counterpart

$$Q_{\mu\nu;\lambda}^\lambda = Q_{\mu\nu,\lambda}^\lambda - Q_{\mu\xi}^\lambda \Gamma_{(\lambda\nu)}^\xi - Q_{\xi\nu}^\lambda \Gamma_{(\mu\lambda)}^\xi + Q_{\mu\nu}^\xi \Gamma_{(\xi\lambda)}^\lambda,\tag{19}$$

$$Q_{\mu\nu}^\lambda = \Gamma_{[\mu\nu]}^\lambda + \frac{1}{3}\delta_\mu^\lambda \Gamma_\nu - \frac{1}{3}\delta_\nu^\lambda \Gamma_\mu, \quad Q_{\mu\lambda}^\lambda = 0\tag{20}$$

as

$$\begin{aligned}2\kappa I &= s^{\mu\nu} \left[R_{\mu\nu} - Q_{\mu\xi}^\lambda Q_{\lambda\nu}^\xi + \left(a + \frac{1}{3}\right) \Gamma_\mu \Gamma_\nu \right] + a^{\mu\nu} \left[Q_{\mu\nu;\lambda}^\lambda + \left(b - \frac{1}{6}\right) [\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}] \right] \\ &\equiv s^{\mu\nu} \left[R_{\mu\nu} - Q_{\mu\xi}^\lambda Q_{\lambda\nu}^\xi + x \Gamma_\mu \Gamma_\nu \right] + a^{\mu\nu} [Q_{\mu\nu;\lambda}^\lambda - y(\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu})]\end{aligned}\tag{21}$$

with $x = a + \frac{1}{3}$, $y = -(b - \frac{1}{6})$. Then using the action principle $\delta I = \delta \int \mathcal{L} d^4x = 0$, and varying $s^{\mu\nu}$ and $a^{\mu\nu}$ independently while keeping the connections fixed give the field equations

$$R_{\mu\nu} - Q_{\mu\xi}^\lambda Q_{\lambda\nu}^\xi + x \Gamma_\mu \Gamma_\nu = R_{\mu\nu} - \Gamma_{[\mu\xi]}^\lambda \Gamma_{[\lambda\nu]}^\xi + a \Gamma_\mu \Gamma_\nu = 0,\tag{22}$$

$$Q_{\mu\nu;\lambda}^\lambda - y(\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}) = 0.\tag{23}$$

Notice that torsion contributes to the first equation (22) (gravity) but equation (23) has no contribution from the symmetrical part. If one sets $a = 0$, one obtains equations that are projective and transposition invariant. We will assume this to be the case to start with. Thus, there is only one free parameter in the theory, namely b which the symmetries permit. We will see in what follows that this is the optimal formulation that is consistent with EEP. Once the matter Lagrangian is introduced, projective symmetry will be broken, and a term proportional to $\Gamma_\mu \Gamma_\nu$ will occur in eqn. (22) with a coefficient determined by the strength a of the symmetry violation. Note that *the symmetries of the theory do not require Γ_μ to vanish*. The consequences will be explored in the following.

The gravitational sector

Since $\Gamma_\mu = \Gamma_{[\mu\lambda]}^\lambda$, one can write

$$\Gamma_{[\mu\xi]}^\lambda = \frac{1}{3} (\Gamma_\mu \delta_\xi^\lambda - \Gamma_\xi \delta_\mu^\lambda)\tag{24}$$

and hence

$$\Gamma_{[\mu\xi]}^\lambda \Gamma_{[\lambda\nu]}^\xi = -\frac{1}{3} \Gamma_\mu \Gamma_\nu.\tag{25}$$

In the presence of matter eqn. (22) can therefore be written as

$$R_{\mu\nu} + x \Gamma_\mu \Gamma_\nu = \kappa T_{\mu\nu}\tag{26}$$

This equation implies that $R = -x\Gamma + \kappa T$ where $R = g^{(\mu\nu)}R_{\mu\nu}$, $T = g^{(\mu\nu)}T_{\mu\nu}$, $\Gamma = g^{(\mu\nu)}\Gamma_\mu\Gamma_\nu$ and hence it can be rewritten in the form

$$G_{\mu\nu} + g_{(\mu\nu)}\lambda = \kappa T_{\mu\nu} - x\Gamma_\mu\Gamma_\nu. \quad (27)$$

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{(\mu\nu)}R, \\ \lambda &= \frac{1}{2}R = \frac{1}{2}(\kappa T - x\Gamma). \end{aligned} \quad (28)$$

With the choice (24) the traceless torsion $Q^\lambda_{\mu\nu} = 0$ and eqn.(23) reduces to

$$\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu} = 0. \quad (29)$$

Hence, Γ_μ is an irrotational pseudovector. As we will see in the section on electromagnetism, Γ_0 is essentially a magnetic monopole density, and hence there is a Dirac string and space is not simply connected. Let $S = \mathbb{R}^3 \setminus \{(0, 0, z \leq 0) | z \in \mathbb{R}\}$ be the usual 3-dimensional space with the negative z -axis removed. Then the curl-free vector $\mathbf{\Gamma} = -\nabla\Phi$, $\nabla^2\Phi = 0$ has vortex solutions $\mathbf{\Gamma} = \mathbf{e}_\phi/r$ where \mathbf{e}_ϕ is a unit vector, and the integral over a unit circular path C enclosing the origin is

$$\oint_C \mathbf{\Gamma} \cdot \mathbf{e}_\phi d\phi = \pm 2\pi. \quad (30)$$

Hence, $\mathbf{\Gamma}$ is quantized in units of 2π though it is still conservative in every subregion of S that does not include the origin and the Dirac string. Similarly, there will be quantized vortices in the (x, t) , (y, t) planes. The implications of all this in classical optics and weak gravitational fields will be elaborated in separate papers to follow [23].

The equations of connection

In order to derive the equations connecting the metric g and the connections Γ , one can use a generalized variation (as shown in the Appendix (Part 2)) to derive the equations

$$g^{\mu\nu}_{,\lambda} + g^{\mu\alpha}\Gamma'_{\lambda\alpha}{}^\nu + g^{\alpha\nu}\Gamma'_{\alpha}{}^\mu{}_\lambda = g^{\mu\nu}P_\lambda \quad (31)$$

where

$$\begin{aligned} \Gamma'_{\lambda\alpha}{}^\nu &= \Gamma_{(\lambda\alpha)}^\nu + \frac{1}{4\sqrt{-g}} (g_{[\lambda\beta]}k^\beta\delta_\alpha^\nu + \lambda \leftrightarrow \alpha) \\ &+ \frac{1}{4\sqrt{-g}} (g_{[\lambda\beta]}k^\beta\delta_\alpha^\nu - \lambda \leftrightarrow \alpha) + Q_{\lambda\alpha}{}^\nu + \frac{1}{\sqrt{-g}}(g_{\lambda\beta}k^\beta\delta_\alpha^\nu - g_{\alpha\beta}k^\beta\delta_\lambda^\nu), \end{aligned} \quad (32)$$

$$\begin{aligned} \Gamma'_{\alpha}{}^\mu{}_\lambda &= \Gamma_{(\alpha\lambda)}^\mu + \frac{1}{4\sqrt{-g}} (g_{[\lambda\beta]}k^\beta\delta_\alpha^\mu + \lambda \leftrightarrow \alpha) \\ &+ \frac{1}{4\sqrt{-g}} (g_{[\lambda\beta]}k^\beta\delta_\alpha^\mu - \lambda \leftrightarrow \alpha) + Q_{\alpha}{}^\mu{}_\lambda + \frac{1}{\sqrt{-g}}(g_{\beta\alpha}k^\beta\delta_\lambda^\mu - g_{\lambda\beta}k^\beta\delta_\alpha^\mu) \end{aligned} \quad (33)$$

are the new connections and

$$P_\lambda = \frac{1}{2\sqrt{-g}} \left(\frac{g_{,\lambda}}{g} + 3g_{[\lambda\beta]}k^\beta \right). \quad (34)$$

These new connections Γ' are clearly not metric compatible. One can define the tensors

$$\begin{aligned}
K_{\lambda\alpha}^{\nu} &= \frac{1}{4\sqrt{-g}} (g_{[\lambda\beta]} k^{\beta} \delta_{\alpha}^{\nu} + \lambda \leftrightarrow \alpha) + \mathcal{K}_{\lambda\alpha}^{\nu}, \\
\mathcal{K}_{\lambda\alpha}^{\nu} &= \frac{1}{4\sqrt{-g}} (g_{[\lambda\beta]} k^{\beta} \delta_{\alpha}^{\nu} - \lambda \leftrightarrow \alpha) + Q_{\lambda\alpha}^{\nu} + \frac{1}{\sqrt{-g}} (g_{\lambda\beta} k^{\beta} \delta_{\alpha}^{\nu} - g_{\alpha\beta} k^{\beta} \delta_{\lambda}^{\nu}), \\
K_{\alpha\lambda}^{\mu} &= \frac{1}{4\sqrt{-g}} (g_{[\lambda\beta]} k^{\beta} \delta_{\alpha}^{\mu} + \lambda \leftrightarrow \alpha) + \mathcal{K}_{\alpha}^{\mu}{}_{\lambda}, \\
\mathcal{K}_{\alpha}^{\mu}{}_{\lambda} &= \frac{1}{4\sqrt{-g}} (g_{[\lambda\beta]} k^{\beta} \delta_{\alpha}^{\mu} - \lambda \leftrightarrow \alpha) + Q_{\alpha}^{\mu}{}_{\lambda} + \frac{1}{\sqrt{-g}} (g_{\beta\alpha} k^{\beta} \delta_{\lambda}^{\mu} - g_{\lambda\beta} k^{\beta} \delta_{\alpha}^{\mu}),
\end{aligned} \tag{35}$$

where \mathcal{K} s are called contorsion tensors.

Since the path-free notion of parallelism must hold locally, i.e. in the infinitesimal neighbourhood of every point on the manifold with coordinates, it is sufficient to impose the metric compatibility condition locally. Then, $P_{\lambda} = 0$,

$$g_{,\lambda}^{\mu\nu} + g^{\mu\alpha} \Gamma'_{\lambda\alpha}{}^{\nu} + g^{\alpha\nu} \Gamma'_{\alpha}{}^{\mu}{}_{\lambda} = 0, \tag{36}$$

and it follows from eqn. (34) that

$$3g_{[\lambda\beta]} k^{\beta} = -\frac{g_{,\lambda}}{g}. \tag{37}$$

The electromagnetic sector

Using the variational principle, it can also be shown (see Appendix (Part 2)) that the relevant part of the Lagrangian density for electrodynamics is

$$\mathcal{L}_{em} = \frac{1}{2\kappa} [y a_{,\lambda}^{\mu\nu} (\Gamma_{\mu} \delta_{\nu}^{\lambda} - \Gamma_{\nu} \delta_{\mu}^{\lambda}) + x s^{\mu\nu} \Gamma_{\mu} \Gamma_{\nu}], \tag{38}$$

and varying it w.r.t. Γ_{μ} , one gets

$$a_{;\nu}^{\mu\nu} = \theta s^{\mu\nu} \Gamma_{\nu} = \frac{1}{2} \theta \sqrt{-g} \Gamma^{\mu}, \quad \theta = -\frac{x}{y}, \tag{39}$$

$$\Gamma_{(\lambda\alpha)}^{\alpha} = \frac{1}{2} \left(\frac{g_{,\lambda}}{g} + g_{[\lambda\beta]} k^{\beta} \right). \tag{40}$$

Eqn. (39) can be written in the form

$$\tilde{F}_{;\nu}^{\mu\nu} = l^{\mu} \tag{41}$$

with

$$\tilde{F}^{\mu\nu} = \zeta a^{\mu\nu}, \tag{42}$$

$$l^{\mu} = \frac{1}{2} \theta \sqrt{-g} \zeta \Gamma^{\mu}, \tag{43}$$

where the constant ζ has the dimension of kg/C. The most natural choice would be $\zeta = 1/\sqrt{4\pi\epsilon_0 G}$ where G is the gravitational constant and ϵ_0 is the absolute permittivity of space. Its current value is $1.16 \times 10^{10} \text{ kg.C}^{-1}$. There is, as we will see, astrophysical evidence that the value of the scale parameter ζ is indeed $1/\sqrt{4\pi\epsilon_0 G}$.

It follows from eqn. (41) that $l^\mu_{,\mu} = 0$. Since l^μ is a pseudovector, the field $\tilde{F}^{\mu\nu}$ is a pseudo tensor. Defining the fields

$$\tilde{F}^{0i} = -B^i, \quad \tilde{F}^{ij} = \frac{1}{c}\epsilon^{ijk}E_k, \quad (44)$$

one gets from (41) and the definition $l^\mu = (-\mu_0\rho_m, -\mathbf{j}_m/c\epsilon_0)$ the equations

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{\epsilon_0}\mathbf{j}_m, \quad \nabla \cdot \mathbf{B} = \mu_0\rho_m. \quad (45)$$

Hence, by comparison with electrodynamics, we can interpret l^μ as a *magnetic current density* and $\tilde{F}_{\mu\nu}$ as the *dual of the electromagnetic field*. The magnetic charge is conserved because the divergence of the magnetic current l^μ vanishes identically. Eqn. (29) imposes a certain constraint on this magnetic current which will be explored elsewhere [23].

Equations (41), and hence also equations (45), can be written in the form

$$F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = \epsilon_{\mu\nu\lambda\rho}l^\rho, \quad (46)$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}, \quad (47)$$

where $\epsilon_{\mu\nu\lambda\rho}$ is the Levi Civita tensor density with components ± 1 . This equation reduces to the well known Bianchi identity in electrodynamics only if $\Gamma^\mu = 0$.

It can be easily checked (using the relations (10)) that the electric current density j^μ in this theory is given by

$$j^\mu = \frac{1}{3!}\epsilon^{\mu\nu\lambda\rho}(\tilde{F}_{\nu\lambda,\rho} + \tilde{F}_{\lambda\rho,\nu} + \tilde{F}_{\rho\nu,\lambda}) \quad (48)$$

$$= \frac{1}{3!}\zeta\epsilon^{\mu\nu\lambda\rho}(a_{\nu\lambda,\rho} + a_{\lambda\rho,\nu} + a_{\rho\nu,\lambda}) \quad (49)$$

$$= F^{\mu\nu}_{,\nu}. \quad (50)$$

Writing $j^\mu = (-\rho_q/c\epsilon_0, -\mu_0\mathbf{j}_q)$ and using the definitions

$$F^{0i} = -\frac{E^i}{c}, \quad F^{ij} = -\epsilon^{ijk}B_k, \quad (51)$$

one then obtains

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0\mathbf{j}_q, \quad \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho_q. \quad (52)$$

These equations can also be written in the form

$$\tilde{F}_{\mu\nu,\lambda} + \tilde{F}_{\nu\lambda,\mu} + \tilde{F}_{\lambda\mu,\nu} = \epsilon_{\mu\nu\lambda\rho}j^\rho. \quad (53)$$

It is clear from this that the dual fields \tilde{F} do not satisfy the standard Bianchi identity because $j^\mu \neq 0$. The Maxwell equations thus acquire a new geometric significance. (The matrix representations of the fields $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ used in this paper are given at the end of the Appendix.)

Thus, the full set of Maxwell's equations in the presence of electric and magnetic currents are equivalently described by one of the following combinations of equations: eqns (45) and (52); or eqns (41) and (50); or eqns (46) and (53); or eqns (41) and (53); or eqns (46) and (50).

The equations (45) and (52) are together invariant under the generalized (Heaviside) duality transformations [24]

$$\begin{aligned}\mathbf{E} &\rightarrow c\mathbf{B}, \\ c\mathbf{B} &\rightarrow -\mathbf{E}, \\ (\rho_q, \mathbf{j}_q) &\rightarrow (\rho_m, \mathbf{j}_m), \\ (\rho_m, \mathbf{j}_m) &\rightarrow (-\rho_q, -\mathbf{j}_q).\end{aligned}\tag{54}$$

This symmetry is therefore a *consequence* of the unified theory. One can define potentials in the usual way through the relations

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,\tag{55}$$

but they turn out to be singular in this theory. According to this definition, a static magnetic field due to a charge g is given by

$$\vec{B} = g \frac{\mathbf{r}}{r^3} = \nabla \times \mathbf{A}.\tag{56}$$

But eqn. (45) contradicts this. It is well known that the solution is the famous Dirac potential [27] which can be written in spherical polar coordinates as

$$\mathbf{A}_\phi = \frac{g}{r} \tan \frac{\theta}{2} \hat{\phi}, \quad A_r = A_\theta = 0\tag{57}$$

whose solution gives

$$\mathbf{B}_r = g \frac{\mathbf{r}}{r^3}, \quad \mathbf{B}_\phi = \mathbf{B}_\theta = 0.\tag{58}$$

This potential is singular along the negative z axis characterized by $\theta = \pi$, called the Dirac string which is a semi-infinite line of magnetic dipoles ending in a monopole at the origin. In the field theory under consideration, the potential will be non-holomorphic in general, and instead of a single monopole, there will be a magnetic density at the origin. Everywhere other than where the potential is non-holomorphic, it will be the standard electromagnetic potential. Hence, the potential due to a magnetic moment $\vec{\mu}$ far from the origin is given by the standard expression

$$A_i = \frac{\mu_0 c}{4\pi} \frac{(\vec{r} \times \vec{\mu})_i}{r^3} = - \int E_i dx^0.\tag{59}$$

Since the potential is gauge dependent, its singularity can be chosen to take any convenient form.

We will discuss the classical ‘quantization’ (in the sense of discretization) of charge and angular momentum in classical optics in an accompanying paper [23].

Einstein’s Equivalence Principle

Let us now see if this unified theory incorporates EEP. Although the connections in the theory are non-symmetric, the geodesic of a test particle in a coordinate frame is determined in this theory by

$$\begin{aligned}a^\lambda &= \frac{d^2 x^\lambda}{ds^2} = -\Gamma'^\lambda_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \\ &= - \left(\Gamma^\lambda_{(\mu\nu)} + \frac{1}{8} (a_{\mu\beta} k^\beta \delta_\nu^\lambda + \mu \leftrightarrow \nu) \right) u^\mu u^\nu \\ &\equiv - (\Gamma^\lambda_{(\mu\nu)} + \bar{\Gamma}^\lambda_{(\mu\nu)}) u^\mu u^\nu\end{aligned}\tag{60}$$

because the antisymmetric part of the connection does not contribute to the geodesic. Thus, the acceleration is universal and consists of two components: the first component is the familiar Einstein gravitational acceleration and the second component is an additional *gravitational* acceleration that is induced by the antisymmetric part $a_{\mu\nu}$ of the metric tensor density *and* the vector $k^\beta = \theta s^{\beta\nu} \Gamma_\nu$.

Since $(\Gamma_{(\mu\nu)}^\lambda + \bar{\Gamma}_{(\mu\nu)}^\lambda)$ is symmetric in the lower indices, the Weyl theorem [28] guarantees that it can be made to vanish in the neighbourhood of a point on the manifold by a coordinate transformation, and hence all observable effects of gravity, including those originating in electromagnetism, can be transformed away in that neighbourhood by a coordinate transformation. This can be explicitly demonstrated as follows.

First note that combining eqns. (37) and (40), one has locally

$$\Gamma_{(\lambda\alpha)}^\alpha = g_{[\lambda\beta]} k^\beta. \quad (61)$$

Now, let p be a point on the manifold $\mathcal{M}(\Gamma, g)$ and let $(U, \chi = (x^\mu))$ and $(U', \chi' = (x'^\mu))$ be two intersecting local charts with $p \in U$, $x^\mu(p) = 0$, $x'^\mu(p) = 0$ and

$$x^\mu = x'^\mu - \frac{1}{2} \Gamma_{(\nu\rho)}^\mu x'^\nu x'^\rho \quad (62)$$

because torsion does not contribute to this change of coordinates. Writing $g_{\mu\nu}(p) = \eta_{\mu\nu} + \dots$ and remembering that $\Gamma_{(\mu\nu)}^\lambda(p)$ can be made to vanish by Weyl's theorem, one gets on using equations (36) and (61),

$$\begin{aligned} g'^{\mu\nu}(x'^\lambda) &= \eta^{\mu\nu} + g'^{\mu\nu}_{,\lambda}(p) x'^\lambda + \mathcal{O}(x'^2) \\ &= \eta^{\mu\nu} + \left[-g^{\mu\alpha} \Gamma'_{\lambda\alpha}{}^\nu - g^{\alpha\nu} \Gamma'_{\alpha}{}^\mu{}_\lambda \right]_p x'^\lambda + \mathcal{O}(x'^2) \end{aligned} \quad (63)$$

$$= \eta^{\mu\nu} - \left[g^{\mu\nu} \frac{1}{4\sqrt{-g}} (g_{[\lambda\beta]} k^\beta \delta_\alpha^\nu + \lambda \leftrightarrow \alpha) \right]_p x'^\lambda + \left[-\eta^{(\mu\alpha)} \mathcal{K}_{[\lambda\alpha]}{}^\nu - \eta^{(\alpha\nu)} \mathcal{K}_{[\alpha}{}^\mu{}_{\lambda]} \right]_p x'^\lambda + \mathcal{O}(x'^2)$$

$$\equiv \eta^{\mu\nu} \left[1 + \frac{1}{4\sqrt{-g}} (\Gamma_{(\lambda\delta)}^\delta \delta_\alpha^\nu + \Gamma_{(\alpha\delta)}^\delta \delta_\lambda^\nu) \right]_p x'^\lambda - [(\mathcal{K}_\lambda{}^{\mu\nu} + \mathcal{K}^{\nu\mu}{}_\lambda)]_p x'^\lambda + \mathcal{O}(x'^2) \quad (64)$$

$$= \eta^{\mu\nu} + \mathcal{O}(x'^2). \quad (65)$$

Thus, the additional gravity induced by electromagnetism can also be transformed away locally.

Thus, both gravity and electromagnetism are geometric structures of the theory, gravity corresponding to the symmetric Riemannian part of the manifold and electromagnetism to its anti-symmetric part together with $\Gamma^\mu \neq 0$ which implies a magnetic current density. This ensures that the acceleration of a charged test particle by a locally homogeneous gravitational field produced by 'electro-gravity' is physically indistinguishable from that of a free test particle at rest in a comoving coordinate system, which is the physical content of EEP.

4 Some Predictions

(i) Spherically symmetric and static solution

Outside a spherically symmetric body of mass M , the gravitational field equation is given by eqn. (26) with $T_{\mu\nu} = 0$. A spherically symmetric and static solution of the equation requires the line element to be given by

$$ds^2 = -c^2 e^{2m} dt^2 + e^{2n} dr^2 + r^2 d\phi^2 + r^2 \sin^2 \phi d\theta^2 \quad (66)$$

where m and n are functions of r , not t . Thus, all off-diagonal elements of $g_{\mu\nu}$ vanish and the diagonal elements are all time independent. In order to calculate the elements of the Ricci tensor $R_{\mu\nu}$, one must first calculate the Christoffel symbols from $g_{\mu\nu}$ by using the connection between them, which in this theory is given by eqn. (36). Since the off-diagonal elements of $g_{\mu\nu}$ vanish, all $g_{[\mu\nu]} = 0$ and eqn. (36) reduces to

$$g_{,\lambda}^{(\mu\nu)} + g^{(\mu\alpha)} (\Gamma'_{(\lambda\alpha)}^\nu + \mathcal{K}_{\lambda\alpha}^\nu) + g^{(\alpha\nu)} (\Gamma'_{(\alpha\lambda)}^\mu + \mathcal{K}_{\alpha\lambda}^\mu) = 0, \quad (67)$$

Interchanging μ and ν , we get

$$g_{,\lambda}^{(\mu\nu)} + g^{(\nu\alpha)} (\Gamma'_{(\lambda\alpha)}^\mu + \mathcal{K}_{\lambda\alpha}^\mu) + g^{(\alpha\mu)} (\Gamma'_{(\alpha\lambda)}^\nu + \mathcal{K}_{\alpha\lambda}^\nu) = 0. \quad (68)$$

Adding these two equations and using the antisymmetry of \mathcal{K} in the lower indices, one has

$$g_{,\lambda}^{(\mu\nu)} + g^{(\mu\alpha)} \Gamma'_{(\alpha\lambda)}^\nu + g^{(\nu\alpha)} \Gamma'_{(\alpha\lambda)}^\mu = 0, \quad (69)$$

from which it follows (on cyclically varying the free indices and summing)) that

$$\Gamma'_{(\mu\nu)}^\lambda = \frac{1}{2} g^{(\lambda\beta)} (g_{(\mu\beta),\nu} + g_{(\nu\beta),\mu} - g_{(\mu\nu),\beta}) = 0 \quad (70)$$

which is the standard expression in Riemannian geometry. Putting $\Gamma_i = 0$ for a static solution and following the standard procedure [29], one obtains the equations

$$R_{00} = e^{2m-2n} \left(-m'' + m'n' - m'^2 - \frac{2m'}{r} \right) = -x\Gamma_0^2 \neq 0, \quad (71)$$

$$R_{11} = \left(m'' - m'n' + m'^2 - \frac{2n'}{r} \right) = 0, \quad (72)$$

$$R_{22} = e^{-2n} (1 + m'r - n'r) - 1 = 0, \quad (73)$$

$$R_{33} = [e^{-2n} (1 - n'r + m'r) - 1] \sin^2\phi = 0, \quad (74)$$

where the prime denotes a derivative with respect to r . It follows from (71) and (72) that

$$e^{2m-2n} \left(\frac{dm}{dr} + \frac{dn}{dr} \right) = \frac{x}{2} r \Gamma_0^2. \quad (75)$$

Integrating with respect to r and remembering that as $r \rightarrow \infty$, $m \rightarrow 0$ and $n \rightarrow 0$ to ensure an asymptotically flat metric, one gets

$$e^{-2n} - e^{2m} = x \int_0^\infty r \Gamma_0^2 dr. \quad (76)$$

Since m and n are very small in the asymptotic region, one can write $e^{2m} \simeq 1 + 2m$ and $e^{-2n} \simeq 1 - 2n$, and hence we have

$$m + n = -\frac{x}{2} \int_0^\infty r \Gamma_0^2 dr \equiv -\beta_\infty \neq 0 \quad (77)$$

where β_∞ is a constant. It follows from this and eqn. (73) that $n' = -m'$ and hence

$$e^{2(m+\beta_\infty)} (1 + 2m'r) = 1. \quad (78)$$

Since

$$\frac{d}{dr} (r e^{2(m+\beta_\infty)}) = e^{2(m+\beta_\infty)} (1 + 2m'r) = 1, \quad (79)$$

one gets on integrating with respect to r

$$e^{2m} = \left(1 + \frac{\alpha}{r}\right) e^{-2\beta_\infty} \quad (80)$$

where α is an integration constant. Therefore,

$$e^{2n} = e^{-2(\beta_\infty+m)} = \left(1 + \frac{\alpha}{r}\right)^{-1}. \quad (81)$$

Choosing $\alpha = -2GM/c^2$, one obtains

$$ds^2 = -c^2 \left(1 - \frac{2GM}{c^2 r}\right) e^{-2\beta_\infty} dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\phi^2 + r^2 \sin^2 \phi d\theta^2, \quad (82)$$

which reduces to the Schwarzschild metric in the limit $\Gamma_0 = 0$. Notice that even in the absence of the conventional stress-energy tensor $T_{\mu\nu}$ (in this case $T_{00} = \rho c^2 = 0$ and hence $M = 0$) which is the source of Einstein gravity, the metric is modified—the nonlocally measured speed of light $c' = c e^{-\beta_\infty}$ is different from its local value c , or equivalently, the time interval is changed from dt to $e^{-\beta_\infty} dt$. Eqn. (77) shows that $\beta_\infty > 0$ if the arbitrary parameter $x (= a + \frac{1}{3}) > 0$. Thus, $c' < c$ requires that $x > 0$.

Notice that

$$1 - \frac{2GM}{c^2 r} = 1 + \frac{2\phi}{c^2}, \quad (83)$$

$$\phi = -\frac{GM}{r}, \quad (84)$$

ϕ being the gravitational potential of the mass M at the centre of the body. Hence, defining $\bar{\phi} = -\beta_\infty c^2$, we have

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right) e^{2\bar{\phi}/c^2} dt^2 + \dots \quad (85)$$

$$= \left(1 + \frac{2\phi}{c^2} + \frac{2\bar{\phi}}{c^2} + \frac{1}{2!} \frac{4\bar{\phi}^2}{c^4} + \dots\right) dt^2 + \dots \quad (86)$$

For $\bar{\phi}/c^2 \ll 1$, the higher order terms in $\bar{\phi}/c^2$ can be ignored. Thus, in the limit of weak gravity, $\bar{\phi}$ is an additional gravitational potential which *is not due to any additional matter*—it is produced by the torsion pseudovector Γ_μ in the U_4 manifold and is present even when the stress-energy tensor of matter is absent. It plays a dual role—it produces the magnetic current density in the electromagnetic sector and an additional gravitational potential in the gravity sector. It is the agent that links gravity and electromagnetism.

Since $\bar{\phi}$ is a constant, in analogy with Newtonian gravity it is possible to write it as

$$\bar{\phi} = -\frac{GM_{\text{eff}}(r)}{r}, \quad M_{\text{eff}}(r) \sim r, \quad (87)$$

where $M_{\text{eff}}(r)$ is an effective mass. For a spherical galaxy having an effective mass of density $\rho_{\text{eff}}(r)$, the effective mass inside a radius r is

$$M_{\text{eff}}(r) = \frac{4}{3} \pi r^3 \rho_{\text{eff}}(r), \quad (88)$$

and hence $\rho_{\text{eff}}(r) \sim r^{-2}$ so that $M_{\text{eff}}(r) \sim r$. It follows at once from this that the velocity of a test particle (a star) at a distance r from the centre of such a galaxy is

$$v = \sqrt{\frac{GM_{\text{eff}}(r)}{r}} = \text{constant}. \quad (89)$$

Thus, *the theory predicts flat rotation curves of stars at large distances from the centre of such a galaxy.* This is, of course, true only in the weak gravity limit. Eqn. (86) predicts higher order corrections. Since such a galaxy is not made of ordinary matter, it must be *non-luminous*.

Let $\Gamma_0^2(r)$ be a bell-shaped function like $\Gamma_0^2(0)e^{-\mu r^2}$, $\mu > 0$. Then,

$$\beta_\infty = \frac{x\Gamma_0^2(0)}{2} \int_0^\infty r e^{-\mu r^2} dr = \frac{x\Gamma_0^2(0)}{4\mu} = -\frac{\bar{\phi}}{c^2}. \quad (90)$$

and

$$\beta(r) = \frac{x\Gamma_0^2(0)}{2} \int_0^r r e^{-\mu r^2} dr = \beta_\infty [1 - e^{-\mu r^2}] = -\frac{\bar{\phi}(r)}{c^2}. \quad (91)$$

Thus, there is a gravitational potential $\bar{\phi}(r)$ around a spherical static body whose asymptotic value is $\bar{\phi}$. One can interpret this as a dark halo of gravitational tidal field acting like a weak gravitational lens.

Since $\Gamma = \Gamma^\nu \Gamma_\nu = \Gamma_0^2$ when $\Gamma_i = 0$ and $T = 0$ for the vacuum solution, the value of the scalar field λ , defined by eqn. (28), at the origin determines the constant β_∞ :

$$\beta_\infty = \frac{1}{2\mu} \lambda(0). \quad (92)$$

(ii) The Robertson-Walker metric and cosmology

The metric of a homogeneous and isotropic universe is given by the metric [30]

$$ds^2 = -c^2 dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] \quad (93)$$

where $a(t)$ is the scale factor and $k = \pm 1$ or 0 . The modified Einstein equations (26) shows that even when $T_{\mu\nu} = 0$, there is a stress-energy tensor

$$T_{\mu\nu}^{(0)} = -x\Gamma_\mu \Gamma_\nu, \quad (94)$$

$$T_{ii}^{(0)} = -x\Gamma_i^2 \equiv -\frac{x\gamma}{\kappa} \quad (95)$$

with $\gamma = \kappa\Gamma_1^2 = \kappa\Gamma_2^2 = \kappa\Gamma_3^2$ to ensure isotropy. The modified Friedmann equations are

$$\frac{\dot{a}^2}{a^2} = \frac{\kappa}{3}\rho c^2 - \frac{kc^2}{a^2} + \frac{x\gamma}{3\kappa}, \quad (96)$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho c^2 + 3p) + \frac{x\gamma}{3\kappa}. \quad (97)$$

The first of these equations can be written in the form

$$\begin{aligned} \frac{\dot{a}^2}{a^2} &= \frac{\kappa}{3}\rho c^2 - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} + \dots \\ \Lambda &= \frac{x\gamma_0}{\kappa c^2} \end{aligned} \quad (98)$$

where Λ is the cosmological constant, γ_0 is the zero-mode of the field γ , and the dots represent the higher modes of γ . The equation of state for the zero mode of the field γ is $\rho c^2 = -p = x(\gamma)_0/\kappa$, i.e. $w = -1$ and represents ‘dark energy’ [31]. When ρ and k are both zero, i.e. the universe is empty and spatially flat, one obtains the de Sitter solution only if the higher order modes of γ are negligible.

Thus, *the scalar field Γ not only determines a cosmological constant accounting for some ‘dark energy’, it also determines an additional gravitational potential of non-material origin responsible for ‘dark matter’-like phenomena* [32]. In addition, the field Γ also predicts perturbations to a homogeneous and isotropic cosmic microwave background radiation caused by its higher order modes.

(iii) Casimir Effect

The Casimir effect is usually explained as the change in the spectrum of zero-point fluctuations of quantum fields brought about by material boundaries [33]. This change requires the occurrence of vacuum energy in the first place, and quantum field theory is the only theory known so far to give rise to it. However, the calculations give a result some 10^{122} times larger than the observed value, though contributions from other fields may lower this value [34]. An alternative to quantum field theory that naturally entails vacuum energy is the unified classical theory under consideration, as shown above. The question that arises therefore is whether this theory can explain the Casimir effect. The answer is positive and a simple argument will now be given. Consider the time-time component of the stress-energy tensor (eqn. (94))

$$T_{00}^{(0)} = -x\Gamma_0^2 \equiv -\frac{x\epsilon}{\kappa}. \quad (99)$$

Consider a *gedanken* Casimir cavity that allows only the zero or lowest-mode of the vacuum field Γ_0 inside. Writing $\epsilon = \epsilon_0 + \epsilon'$ where ϵ_0 is the energy density inside the cavity, let us consider the difference

$$\frac{x}{\kappa}\epsilon' = \frac{x}{\kappa}(\epsilon - \epsilon_0) \neq 0. \quad (100)$$

This is therefore an expression for the Casimir effect. Cosmological observations indicate that ϵ' is extremely small and positive.

(iv) Schuster-Blackett-Wilson Relation

It is well known that for weak gravitational fields one can write $g_{(\mu\nu)} = \eta_{\mu\nu} + h_{\mu\nu}$ where h is a perturbation on the Minkowski metric η , and the gravitational field equations reduce to Maxwell-like forms

$$2F_g^{\mu\nu} = \partial^\mu A_g^\nu - \partial^\nu A_g^\mu, \quad (101)$$

$$\partial_\mu F_g^{\mu\nu} = -4\pi G j^\nu, \quad (102)$$

where $A_\mu^g = (-A_0^g, \mathbf{A}^g)$, $A_0^g = \phi_g$. In analogy with electrodynamics, one can also define the fields \mathbf{E}_g and \mathbf{B}_g by

$$\mathbf{E}_g = -\nabla\Phi_g - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{A}_g\right), \quad (103)$$

$$\mathbf{B}_g = \nabla \times \mathbf{A}_g, \quad (104)$$

with

$$A_i^g = -\frac{1}{2}c^2 h_{(0i)} = -\frac{1}{2}c^2 g_{(0i)} = -2 \int E_{gi} dx^0. \quad (105)$$

These are called respectively the gravitoelectric and gravitomagnetic fields [35]. The main difference with Newtonian gravity is the existence of the gravitomagnetic field \mathbf{B}_g . Some fundamental differences with electrodynamics are reflected in the minus sign in (102) (gravity is always attractive) and factors of 2 because gravitational waves are radiated by quadrupoles.

In the unified theory under consideration the universal dimensional parameter connecting the electromagnetic field to the metric components (and therefore to the gravitational field) being $\zeta|\hat{a}|$ (eqn. (42)), the ratio of the gravitational and electromagnetic accelerations E_{gi} and E_i is fixed to be

$$\frac{E_{gi}}{E_i} = \frac{1}{\zeta}. \quad (106)$$

It follows from eqns (59) and (105) therefore that

$$\frac{A_i^g}{A_i} = \frac{2}{\zeta}. \quad (107)$$

Now, the gravitomagnetic vector potential A_i^g at a large distance from a small rotating body of angular momentum \vec{J} is given by [36, 37]

$$A_i^g = -\frac{G}{c} \frac{(\vec{r} \times \vec{J})_i}{r^3}. \quad (108)$$

and the electromagnetic potential at a large distance from a magnetic moment $\vec{\mu}$ is given by (59). If θ_J and θ_μ are the angles between the radius vector \vec{r} and \vec{J} and \vec{r} and $\vec{\mu}$ respectively, and $\beta = (\sin\theta_J/\sin\theta_\mu)$, eqn. (107) predicts that the gyromagnetic ratio of the body is given by

$$\gamma = \frac{|\vec{\mu}|}{|\vec{J}|} \simeq \frac{1}{2\zeta} \beta = \frac{\sqrt{G}}{2\sqrt{k}} \beta = 4.3 \times 10^{-11} \beta \text{ C/kg}, \quad (109)$$

where $k = 1/4\pi\epsilon_0$. This is the empirical Schuster-Blackett-Wilson relation if the factor β is of order unity. This empirical law is valid for an amazingly wide variety of astronomical bodies [38, 39]. Einstein had proposed a similar relationship in 1924 to account for terrestrial and solar magnetism [40].

An immediate implication of this is that slowly rotating *spherical and electrically neutral bodies generate both gravitational and magnetic fields*. This provides a possible unified theoretical basis of the origin of cosmic magnetic fields that pervade the universe and of the intense magnetic fields near rotating black holes, connected with quasars and gamma-ray bursts for whose origin the Schuster-Blackett-Wilson relation has been used as a mechanism for non-minimal gravitational-electromagnetic coupling (NMGEC) [41, 42, 43]. Furthermore, the unified theory unequivocally predicts the presence of primordial magnetic fields and curlless magnetic currents (29) which should have important consequences for CMBR anisotropies [44] and other cosmic phenomena.

5 Concluding Remarks

I have argued that radiation by uniformly accelerated charged particles is a strong indication to look for a geometric unification of gravity and electromagnetism right at the classical level. Such a theory, based on a metric-affine U_4 manifold, has been constructed. It is a variation of the theory proposed by S. N. Bose in 1953, and some of its physical implications have been worked out. The theory differs from the one proposed by Einstein [21] mainly in admitting a non-vanishing torsion pseudovector Γ_μ

which links gravity and electromagnetism. As we have seen, this leads to a modification of Einstein gravity with a cosmological constant which is the zero-mode of the scalar field $\Gamma = \Gamma^\mu \Gamma_\mu$, implying an accelerated expansion of the universe, and hence to a simple understanding of ‘dark energy’ and CMBR perturbations, albeit qualitative at this stage.

A spherically symmetric and static solution of the modified Einstein equation leads to a qualitative understanding of also ‘dark matter’-like phenomena such as non-material and hence non-luminous halos surrounding galaxies, flat rotation curves and weak gravitational lensing. In addition, it also predicts a Casimir Effect without quantum zero-point fluctuations.

The theory predicts Maxwell’s equations with a magnetic current proportional to Γ_μ in addition to an electric current, thus satisfying Heaviside duality, *even in the absence of any matter*. The charges are therefore of a geometric or topological nature, and Maxwell’s equations acquire a new significance.

Thus, the proposed unification of gravity and electromagnetism offers a conceptually simple understanding of a wide range of physical phenomena including both ‘dark matter’ and ‘dark energy’, two of the most outstanding problems in physics today. At the same time it also predicts the long conjectured Schuster-Blackett-Wilson relation for the gyromagnetic ratio of a striking number of rotating astrophysical bodies, opening up the possibility of further investigations into the origin of cosmic magnetic fields and their effects on CMBR anisotropies.

The implications for classical optics and weak gravitational fields have also been worked out and will be elaborated in separate papers to follow.

6 Acknowledgement

The author is grateful to Anirban Mukherjee for helpful discussions and to the National Academy of Sciences, India for a grant.

7 Appendix

Part 1

The generalized Ricci curvature tensor on $\mathcal{M}(\Gamma, g)$ is

$$E_{\mu\nu} = \Gamma_{\mu\nu, \lambda}^\lambda - \Gamma_{\mu\lambda, \nu}^\lambda + \Gamma_{\mu\nu}^\xi \Gamma_{\xi\lambda}^\lambda - \Gamma_{\mu\lambda}^\xi \Gamma_{\xi\nu}^\lambda. \quad (110)$$

By transposition it is converted into

$$\tilde{E}_{\nu\mu} = \Gamma_{\nu\mu, \lambda}^\lambda - \Gamma_{\lambda\mu, \nu}^\lambda + \Gamma_{\nu\mu}^\xi \Gamma_{\lambda\xi}^\lambda - \Gamma_{\lambda\mu}^\xi \Gamma_{\nu\xi}^\lambda. \quad (111)$$

Therefore consider the invariant

$$\begin{aligned}
\frac{1}{2} \left[\bar{g}^{\mu\nu} E_{\mu\nu} + \bar{g}^{\nu\mu} \tilde{E}_{\nu\mu} \right] &= \left[s^{\mu\nu} (E_{\mu\nu} + \tilde{E}_{\nu\mu}) + a^{\mu\nu} (E_{\mu\nu} - \tilde{E}_{\nu\mu}) \right] \\
&= s^{\mu\nu} \left[R_{\mu\nu} - \Gamma_{[\mu\xi]}^\lambda \Gamma_{[\lambda\nu]}^\xi \right] \\
&+ a^{\mu\nu} \left(\Gamma_{[\mu\nu],\lambda}^\lambda - \Gamma_{[\mu\lambda],\nu}^\lambda + \Gamma_{\mu\nu}^\xi \Gamma_{\xi\lambda}^\lambda - \Gamma_{\mu\lambda}^\xi \Gamma_{\xi\nu}^\lambda - \Gamma_{\nu\mu}^\xi \Gamma_{\lambda\xi}^\lambda + \Gamma_{\lambda\mu}^\xi \Gamma_{\nu\xi}^\lambda \right) \\
&= s^{\mu\nu} \left[R_{\mu\nu} - \Gamma_{[\mu\xi]}^\lambda \Gamma_{[\lambda\nu]}^\xi \right] \\
&+ a^{\mu\nu} \left[\Gamma_{[\mu\nu],\lambda}^\lambda - \frac{1}{2} (\Gamma_{\mu,\nu}^\lambda - \Gamma_{\nu,\mu}^\lambda) + Q_{\mu\nu}^\xi \Gamma_{(\xi\lambda)}^\lambda - Q_{\mu\xi}^\lambda \Gamma_{(\lambda\nu)}^\xi - Q_{\xi\nu}^\lambda \Gamma_{(\mu\lambda)}^\xi \right] \\
&= s^{\mu\nu} \left[R_{\mu\nu} - \Gamma_{[\mu\xi]}^\lambda \Gamma_{[\lambda\nu]}^\xi \right] + a^{\mu\nu} \left[Q_{\mu\nu;\lambda}^\lambda - \frac{1}{6} (\Gamma_{\mu,\nu}^\lambda - \Gamma_{\nu,\mu}^\lambda) \right] \tag{112}
\end{aligned}$$

where

$$R_{\mu\nu} = \Gamma_{(\mu\nu),\lambda}^\lambda - \Gamma_{(\mu\lambda),\nu}^\lambda + \Gamma_{(\mu\nu)}^\xi \Gamma_{(\xi\lambda)}^\lambda - \Gamma_{(\mu\lambda)}^\xi \Gamma_{(\xi\nu)}^\lambda, \tag{113}$$

$$Q_{\mu\nu}^\lambda = \Gamma_{[\mu\nu]}^\lambda + \frac{1}{3} \delta_\mu^\lambda \Gamma_\nu - \frac{1}{3} \delta_\nu^\lambda \Gamma_\mu, \tag{114}$$

$$\begin{aligned}
Q_{\mu\nu;\lambda}^\lambda &= Q_{\mu\nu,\lambda}^\lambda - Q_{\mu\xi}^\lambda \Gamma_{(\lambda\nu)}^\xi - Q_{\xi\nu}^\lambda \Gamma_{(\mu\lambda)}^\xi + Q_{\mu\nu}^\xi \Gamma_{(\xi\lambda)}^\lambda \\
&= \Gamma_{[\mu\nu],\lambda}^\lambda + \frac{1}{3} \delta_\mu^\lambda \Gamma_{\nu,\lambda} - \frac{1}{3} \delta_\nu^\lambda \Gamma_{\mu,\lambda} + \dots \\
&= \Gamma_{[\mu\nu],\lambda}^\lambda - \frac{1}{3} (\Gamma_{\mu,\nu}^\lambda - \Gamma_{\nu,\mu}^\lambda) + Q_{\mu\nu}^\xi \Gamma_{(\xi\lambda)}^\lambda - Q_{\mu\xi}^\lambda \Gamma_{(\lambda\nu)}^\xi - Q_{\xi\nu}^\lambda \Gamma_{(\mu\lambda)}^\xi, \\
Q_{\mu\xi}^\lambda Q_{\lambda\nu}^\xi &= \left(\Gamma_{[\mu\xi]}^\lambda + \frac{1}{3} \delta_\mu^\lambda \Gamma_\xi - \frac{1}{3} \delta_\xi^\lambda \Gamma_\mu \right) \left(\Gamma_{[\lambda\nu]}^\xi + \frac{1}{3} \delta_\lambda^\xi \Gamma_\nu - \frac{1}{3} \delta_\nu^\xi \Gamma_\lambda \right) \\
&= \left(\Gamma_{[\mu\xi]}^\lambda \Gamma_{[\lambda\nu]}^\xi - \frac{1}{3} \Gamma_\mu \Gamma_\nu \right). \tag{115}
\end{aligned}$$

Hence,

$$\kappa I = s^{\mu\nu} \left[R_{\mu\nu} - Q_{\mu\xi}^\lambda Q_{\lambda\nu}^\xi + (a + \frac{1}{3}) \Gamma_\mu \Gamma_\nu \right] + a^{\mu\nu} \left(Q_{\mu\nu;\lambda}^\lambda + (b - \frac{1}{6}) [\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}] \right). \tag{116}$$

Part 2

Let

$$\kappa I = H + \frac{dX^\lambda}{dx^\lambda} \tag{117}$$

with

$$\begin{aligned}
X^\lambda &= s^{\mu\nu} \Gamma_{(\mu\nu)}^\lambda - s^{\mu\lambda} \Gamma_{(\mu\nu)}^\nu + a^{\mu\nu} [Q_{\mu\nu}^\lambda - y (\Gamma_\mu \delta_\nu^\lambda - \Gamma_\nu \delta_\mu^\lambda)] \\
H &= -s_{,\lambda}^{\mu\nu} \Gamma_{(\mu\nu)}^\lambda + s_{,\lambda}^{\mu\lambda} \Gamma_{(\mu\nu)}^\nu + s^{\mu\nu} \left(\Gamma_{(\mu\nu)}^\xi \Gamma_{(\xi\lambda)}^\lambda - \Gamma_{(\mu\lambda)}^\xi \Gamma_{(\xi\nu)}^\lambda - Q_{\mu\xi}^\lambda Q_{\lambda\nu}^\xi + x \Gamma_\mu \Gamma_\nu \right) \\
&- a_{,\lambda}^{\mu\nu} [Q_{\mu\nu}^\lambda - y (\Gamma_\mu \delta_\nu^\lambda - \Gamma_\nu \delta_\mu^\lambda)] + a^{\mu\nu} [-Q_{\mu\xi}^\lambda \Gamma_{(\lambda\nu)}^\xi - Q_{\xi\nu}^\lambda \Gamma_{(\mu\lambda)}^\xi + Q_{\mu\nu}^\xi \Gamma_{(\xi\lambda)}^\lambda]
\end{aligned}$$

Thus, H is free of the partial derivatives of $\Gamma_{(\mu\nu)}^\lambda$, $Q_{\mu\nu}^\lambda$ and Γ_μ , and the four-divergence term in the action integral is equal to a surface integral at infinity on which all arbitrary variations are taken to vanish.

Now, it follows from the definition of $Q_{\mu\nu}^\lambda$ that $Q_{\mu\lambda}^\lambda = 0$, and hence all the 24 components of $Q_{\mu\nu}^\lambda$ are not independent. Remembering that these four relations must always hold good in the variations of the elements $\Gamma_{(\mu\nu)}^\lambda, Q_{\mu\nu}^\lambda, \Gamma_\mu$, one can use the method of undetermined Lagrange multipliers k^μ to derive the equations of connection by varying the function

$$H - 2k^\mu Q_{\mu\lambda}^\lambda, \quad (118)$$

namely by requiring

$$\delta \int (H - 2k^\mu Q_{\mu\lambda}^\lambda) d^4x = 0. \quad (119)$$

It is easy to see that variations of H w.r.t $\Gamma_{(\mu\nu)}^\lambda, Q_{\mu\nu}^\lambda$ and Γ_μ give respectively the three equations

$$s_{,\lambda}^{\mu\nu} + s^{\mu\alpha} \Gamma_{(\lambda\alpha)}^\nu + s^{\alpha\nu} \Gamma_{(\alpha\lambda)}^\mu - s^{\mu\nu} \Gamma_{(\lambda\alpha)}^\alpha = -[a^{\mu\alpha} Q_{\lambda\alpha}{}^\nu + a^{\alpha\nu} Q_\alpha{}^\mu{}_\lambda] \quad (120)$$

$$a_{,\lambda}^{\mu\nu} + a^{\mu\alpha} \Gamma_{(\lambda\alpha)}^\nu + a^{\alpha\nu} \Gamma_{(\alpha\lambda)}^\mu - a^{\mu\nu} \Gamma_{(\lambda\alpha)}^\alpha - k^\mu \delta_\lambda^\nu + k^\nu \delta_\lambda^\mu = -[s^{\mu\alpha} Q_{\lambda\alpha}{}^\nu + s^{\alpha\nu} Q_\alpha{}^\mu{}_\lambda] \quad (121)$$

and

$$y a_{,\nu}^{\mu\nu} + x s^{\mu\nu} \Gamma_\nu = 0. \quad (122)$$

It follows from these equations that

$$s_{,\alpha}^{\mu\alpha} + s^{\alpha\beta} \Gamma_{(\alpha\beta)}^\mu + a^{\alpha\beta} Q_{\alpha\beta}^\mu = 0, \quad (123)$$

$$a_{,\nu}^{\mu\nu} = 3k^\mu, \quad (124)$$

which imply

$$k_{,\mu}^\mu = 0. \quad (125)$$

Adding (120) and (121), we get

$$\bar{g}_{,\lambda}^{\mu\nu} + \bar{g}^{\mu\alpha} (\Gamma_{(\lambda\alpha)}^\nu + Q_{\lambda\alpha}{}^\nu) + \bar{g}^{\alpha\nu} (\Gamma_{(\alpha\lambda)}^\mu + Q_\alpha{}^\mu{}_\lambda) - \bar{g}^{\mu\nu} \Gamma_{(\lambda\alpha)}^\alpha - k^\mu \delta_\lambda^\nu + k^\nu \delta_\lambda^\mu = 0, \quad (126)$$

where $\bar{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$ (ref eqn. 10). Multiplying (126) by $\bar{g}_{\mu\nu}$ and using the results

$$\bar{g}^{\mu\nu} \bar{g}_{\mu\lambda} = \delta_\lambda^\nu, \quad \bar{g}^{\mu\nu} \bar{g}_{\lambda\nu} = \delta_\lambda^\mu, \quad Q_{\alpha\lambda}^\lambda = 0, \quad (127)$$

$$g^{\mu\alpha} g_{\alpha\beta} k^\beta = k^\mu, \quad g^{\alpha\nu} g_{\beta\alpha} k^\beta = k^\nu, \quad (128)$$

we first observe that

$$\Gamma_{(\lambda\alpha)}^\alpha = \frac{1}{2} (\bar{g}_{\mu\nu} \bar{g}_{,\lambda}^{\mu\nu} + g_{[\lambda\beta]} k^\beta) = \frac{1}{2} \left(\frac{g_{,\lambda}}{g} + g_{[\lambda\beta]} k^\beta \right). \quad (129)$$

Hence, (126) takes the form

$$g_{,\lambda}^{\mu\nu} + g^{\mu\alpha} \Gamma_{\lambda\alpha}{}^\nu + g^{\alpha\nu} \Gamma_\alpha{}^\mu{}_\lambda = g^{\mu\nu} P_\lambda$$

where

$$\begin{aligned} \Gamma_{\lambda\alpha}{}^\nu &= \Gamma_{(\lambda\alpha)}^\nu + \frac{1}{4\sqrt{-g}} (g_{[\lambda\beta]} k^\beta \delta_\alpha^\nu + \lambda \leftrightarrow \alpha) \\ &+ \frac{1}{4\sqrt{-g}} (g_{[\lambda\beta]} k^\beta \delta_\alpha^\nu - \lambda \leftrightarrow \alpha) + Q_{\lambda\alpha}{}^\nu + \frac{1}{\sqrt{-g}} (g_{\lambda\beta} k^\beta \delta_\alpha^\nu - g_{\alpha\beta} k^\beta \delta_\lambda^\nu), \\ \Gamma_\alpha{}^\mu{}_\lambda &= \Gamma_{(\alpha\lambda)}^\mu + \frac{1}{4\sqrt{-g}} (g_{[\lambda\beta]} k^\beta \delta_\alpha^\mu + \lambda \leftrightarrow \alpha) \\ &+ \frac{1}{4\sqrt{-g}} (g_{[\lambda\beta]} k^\beta \delta_\alpha^\mu - \lambda \leftrightarrow \alpha) + Q_\alpha{}^\mu{}_\lambda + \frac{1}{\sqrt{-g}} (g_{\beta\alpha} k^\beta \delta_\lambda^\mu - g_{\lambda\beta} k^\beta \delta_\alpha^\mu), \\ P_\lambda &= \frac{1}{2\sqrt{-g}} \left(\frac{g_{,\lambda}}{g} + 3g_{[\lambda\beta]} k^\beta \right). \end{aligned} \quad (130)$$

If one imposes the metricity condition $P_\lambda = 0$, it follows from eqn. (129) that

$$\Gamma_{(\lambda\alpha)}^\alpha = -g_{[\lambda\beta]}k^\beta. \quad (131)$$

Finally, we give the matrix forms of the electromagnetic field tensors used:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}, \quad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}$$

References

- [1] W. Pauli, *Theory of Relativity*, translated from German by G. Field, Pergamon Press Ltd. (1958), section 51.
- [2] Paul Zelinsky and Adam Cowl (private communication). See also [14].
- [3] A. Einstein, *Ann. der Phys.* **354** (7), 769-822 (1916).
- [4] M. Born, *Ann. Phys. Lpz.* **30**, 39 (1909).
- [5] G. Nordström, *Proc. Roy. Acad. Amsterdam* **22**, 145-149 (1920). (1920),
- [6] R. P. Feynman, F. B. Morinigo, W. Wagner, B. Hatfield, *Feynman Lectures on Gravitation*, Frontiers in Physics Series, Westview Press (2002).
- [7] R. Becker, *Electromagnetic Fields and Interactions*, Blaisdell Book in the Pure and Applied Sciences, Dover Publications (1982).
- [8] R. T. Hammond, *Elec J of Theoret. Phys.* **7** No. 23, 221-258 (2010).
- [9] H. Bondi and T. Gold, *Proc. R. Soc. London Sect. A* **229**, 416 (1955).
- [10] B. S. De Witt and R. W. Brehme, *Ann. Phys. (N. Y.)*, **9**, 220-259, (1960).
- [11] T. Fulton and F. Rohrlich, *Ann. Phys. (N.Y.)* **9**, 499-517 (1960).
- [12] D. G. Boulware, *Ann. Phys. (N. Y.)* **124**, 169-188 (1980).
- [13] S. Parrott, *Found. Phys.* **32**, 407-440 (2002); //http: arxiv: 9303025 [gr-qc] 2001.
- [14] O. Grøn, Electrodynamics of Radiating Charges, *Adv. in Math. Phys.* **2012**, Article ID 528631, 29 pages, (2012). doi:10.1155/2012/528631 and references therein.
- [15] S. N. Lyle, *Uniformly Accelerating Charged particles: A Threat to the Equivalence Principle*, Fundamental Theories of Physics vol 158, (Springer 2008) and references therein.
- [16] H. F. M. Goenner, ‘On the History of Unified Field Theories’, *Living Reviews in Relativity*, Max Planck Institute for Gravitational Physics (2004).
- [17] See Ref. [1], Note 23.
- [18] A. S. Eddington, *Proc. R. Soc. London, Ser. A* **99**, 104122, (1921).

- [19] A. Einstein, *Sitzungsber. Preuss. Akad. Wiss.* **5**, 32-38, (1923), *ibid* **22**, 414-419, (1925).
- [20] See Ref. [1], Note 23 for other references.
- [21] A. Einstein, *The Meaning of Relativity*, Methuen & Co., London, Sixth Edition, 1956.
- [22] S. N. Bose, *Le Jour de Phys et le Radium* (Paris) **14**, 641-644 (1953).
- [23] P. Ghose and A. Mukherjee, in preparation.
- [24] G. Heaviside, *Phil. Trans. Roy. Soc.(London)* **A 183**, 423 (1893).
- [25] I. Larmor, *Collected Papers*, London (1928);
G. I. Rainich, *Trans. Am. Math. Soc.* **27**, 106 (1925).
- [26] L. Silberstein, *Ann. Phys.* **327**, 579 (1907); **329**, 783 (1907).
I. Bialynicki-Birula and Z. Bialynicka-Birula, *J. Phys. A: Math. Theor.* **46**, 053001 (2013).
- [27] P. A. M. Dirac, *Proc. Roy. Soc.* **A133**, 60 (1931).
- [28] H. Weyl, *Space, Time, Matter*, 4th German edn, Methuen, London (1922).
- [29] D. Simpson, ‘A Mathematical Derivation of the General Relativistic Schwarzschild Metric’, East Tennessee State University (USA) thesis (2007),
<http://www.etsu.edu/cas/math/documents/theses/simpson-thesis.pdf>.
- [30] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, Freeman, San Francisco, (1973).
- [31] M. Li, Xiao-Dong Li, S. Wang, Y. Wang, *Front. of Phys.* **8**, 828 (2013).
- [32] K. Garrett and G. Düda, ‘Dark Matter: A Primer’, *Adv. in Astronomy* **2011**, Article ID 968283 (2011).
- [33] H. B. G. Casimir, *Proc. Kon. Nederland. Akad. Wetensch.* **B51**, 793 (1948).
- [34] J. Martin, *Comptes Rendus Physique* **13**, 566 (2012). arxiv 1205.3365 (2012).
- [35] B. Mashhoon, arXiv:gr-qc/0011014 3 Nov 2000 and references therein; *Proceedings of the XXIII Spanish Relativity Meeting on Reference Frames and Gravitomagnetism*, Edited by J. F. Pascual, -S Anchez, L. Floria, A. San Miguel and F. Vicente, pp. 121-32, (World Scientific, Sigapore) (2001).
- [36] C. Lammerzahl and G. Neugebauer, ‘The Lense-Thirring Effect: From the Basic Notions to the Observed Effects’, *Lecture Notes in Physics* 562, C. Lammerzahl, C. W. F. Francis, and F. W. Hehl (Eds.), pp. 31-51, Springer-Verlag: Berlin Heidelberg (2001).
- [37] J. R. Medina, *Gravitomagnetism (GEM): A Group Theoretic Approach*, Drexel University Thesis (2006); <https://idea.library.drexel.edu/islandora/object/idea:1123>.
- [38] A. Schuster, *Proc. Lond. Phys. Soc.* **24** (1912), 121.
P. M. S. Blackett, *Nature* **159** (1947) 658.
H. A. Wilson, *Proc. Roy.Soc. A* 104, 451-455 (1923).
- [39] Saul-Paul Sirag, *Nature* **278** (1979), 535; *Gravitational Magnetism: an Update*, Vigier III Symposium.

- [40] A. Einstein, *Schw. Naturf. Ges. Verh.* 105 Pt. **2** 1924, 85.
S. Saunders and H. R. Brown, *Philosophy of Vacuum* (1991) (Oxford: Clarendon).
- [41] R. Opher and U. F. Wichowski, *Phys. Rev. Lett.* **78** (1997), 787-790.
- [42] R. S. de Souza and R. Opher, *J. Cosmol. Astropart. Phys.* **02**, (2010), 022.
- [43] R. S. de Souza and R. Opher, *Phys. Lett. B* **705** (2011), 292.
- [44] A. Lewis, *Phys.Rev.* **D70**, 043011 (2004); arXiv: astro-ph/0406096, 2004.